J. Fluid Mech. (1975), *vol.* 69, *part* 1, *pp.* 1–20
Printed in Great Britain

The buckling and stretching of a viscida

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(Received 3 December 1973)

We consider the deformation of a thin thread of viscous liquid (viscida) as its ends are slowly moved together. Equations are deduced which are capable of describing the motion of the thread when the displacement of the axis from a straight line is either on the scale of the thread thickness (problem 1) or on the much larger scale of the thread length (problem 2). In the former case it is shown analytically that an arbitrary initial displacement evolves in such a way that, as the appropriately scaled time τ becomes large, the first mode of the disturbance emerges in a dominant role with an amplitude that is proportional to $\tau^{\frac{1}{2}}$ and independent of the initial amplitude. This provides the initial condition for problem 2, for which a numerical description is obtained.

In addition, we analyse the situation when the ends of the viscida are slowly pulled apart. In this case the high frequency end of the spectrum dominates as an arbitrary disturbance decays.

1. Introduction

The problem of the buckling of a viscoelastic strip has received a great deal of attention and many of its aspects are now well understood. This problem has two natural limits of course: the elastic limit (the problem of the elastica) and the viscous limit. We shall refer to the latter as the problem of the *viscida*, and its study is the subject of the present paper.

The response of layers of viscous fluid to end loads appears to be of geophysical interest, and this provides part of the motivation for the present study. Thus Biot (1964) discussed the plastic deformation of thin layers of rock in the earth's crust by treating the motion as that of a slow viscous fluid. His analysis was restricted to small deformations.

Additional motivation is provided by a brief qualitative experiment described by Taylor (1969). In that experiment, a viscida was floated on mercury and its ends were pushed together. The resulting deformed shape was somewhat reminiscent of the third mode of buckling of an elastica, a result which Taylor explained by noting the analogy between the equations of linear elasticity and those of slow viscous flow. We shall see in the subsequent analysis the extent to which this explanation is, and is not, correct.

The specific problem that we shall consider is that of a two-dimensional viscida, immersed in a vacuum, whose ends are moved. This motion is assumed to be so slow that all the inertia terms may be neglected. In a certain sense, this can be thought of as a model for Taylor's experiment. It also represents a generalization of Biot's work, in the sense that the deformations are not, in general, small.

The foundation of our analysis is the assumption that if the length of the viscida is $O(1)$ then the thickness is $O(\epsilon)$, $\epsilon \ll 1$, so that an asymptotic description is possible. Certain details of the analysis depend on the order of magnitude of the centre-line displacement and calculations are carried out when this is either $O(\epsilon)$ or $O(1)$. The rate of elongation of the viscida plays a role in the former case, but not in the latter.

The motion of the viscida is deduced by formally expanding the solution to the Stokes equations in powers of ϵ. The terms in this expansion satisfy simplified equations which can be integrated, just as the equations of lubrication theory can be integrated. The general solutions contain arbitrary functions of s, the distance measured along the centre-line, and these functions are found by satisfying stress-free conditions at the two free surfaces. There is in general a feedback mechanism, in that the solution cannot be found to a particular order without consideration of higher-order terms in the expansion. Use of integrated equations which represent a global balance of forces and moments substantially simplifies the analysis.

The central result of the above procedure is a partial differential equation for α, the slope of the centre-line, as a function of time and distance from one end. This differential equation contains two unknown functions of time which are essentially determined from integral constraints on α. When the centre-line deviation is small, the equation may be linearized and a complete discussion based on eigenfunction expansions is possible. When α is not small, numerical calculations are necessary. Parts of this analysis are based on the Ph.D. thesis of Nachman (1973).

2. Equations and kinematics

In order to describe the motion of the constituent liquid of the viscida, we choose a curvilinear co-ordinate system based on the instantaneous position of the centre-line. Thus s is measured along the centre-line from the left-hand end of the thread and n is the co-ordinate normal to the centre-line (figure 1). The *exact* equations for an inertialess, incompressible, viscous fluid may then be written in the form

$$\partial u/\partial s + \partial(hv)/\partial n = 0. \tag{2.1a}$$

$$\frac{\partial p_{ss}}{\partial s} + \frac{\partial}{\partial n}(hp_{ns}) + p_{ns}\frac{\partial h}{\partial n} = 0, \tag{2.1b}$$

$$\frac{\partial}{\partial n}(hp_{nn}) + \frac{\partial p_{ns}}{\partial s} - p_{ss}\frac{\partial h}{\partial n} = 0, \tag{2.1c}$$

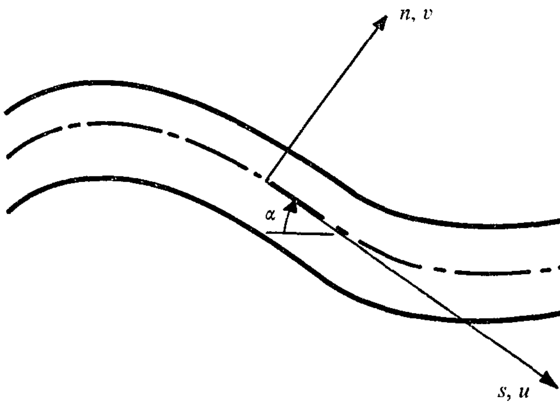

FIGURE 1. Co-ordinate system.

where h is related to K, the curvature of the axis, by

$$h = 1+nK \quad (K = \partial\alpha/\partial s)$$

and the components of the stress tensor are related to the velocity components by means of the constitutive relations

$$p_{ss} = -p+\frac{2\mu}{h}\left(\frac{\partial u}{\partial s}+vK\right), \tag{2.2a}$$

$$p_{nn} = -p+2\mu\,\partial v/\partial n, \tag{2.2b}$$

$$p_{ns} = \mu\left[\frac{1}{h}\frac{\partial v}{\partial s}+h\frac{\partial}{\partial n}\left(\frac{u}{h}\right)\right]. \tag{2.2c}$$

The momentum equations, when written in terms of the velocity components, are

$$-\frac{1}{h}\frac{\partial p}{\partial s}+\mu\frac{\partial}{\partial n}\left\{\frac{1}{h}\left[\frac{\partial}{\partial n}(hu)-\frac{\partial v}{\partial s}\right]\right\} = 0, \tag{2.3a}$$

$$\frac{\partial p}{\partial n}+\frac{\mu}{h}\frac{\partial}{\partial s}\left\{\frac{1}{h}\left[\frac{\partial}{\partial n}(hu)-\frac{\partial v}{\partial s}\right]\right\} = 0. \tag{2.3b}$$

These equations have to be solved subject to certain boundary conditions at the edges of the strip $n = \pm\frac{1}{2}T$, where T is the thickness. In the absence of surface tension the stress tensor vanishes at the edges, whence at $n = \frac{1}{2}T$,

$$p_{nn}(1+\tfrac{1}{2}KT)-\tfrac{1}{2}(\partial T/\partial s)\,p_{ns} = 0, \tag{2.4a}$$

$$p_{ns}(1+\tfrac{1}{2}KT)-\tfrac{1}{2}(\partial T/\partial s)\,p_{ss} = 0, \tag{2.4b}$$

and at $n = -\frac{1}{2}T$,

$$p_{nn}(1-\tfrac{1}{2}KT)+\tfrac{1}{2}(\partial T/\partial s)\,p_{ns} = 0, \tag{2.5a}$$

$$p_{ns}(1-\tfrac{1}{2}KT)+\tfrac{1}{2}(\partial T/\partial s)\,p_{ss} = 0. \tag{2.5b}$$

It is worth noting that, since the inertia terms have been neglected, time does not explicitly appear in the equations or boundary conditions, and in fact plays no role in the analysis until we consider the kinematics of the thread motion.

We shall make use of the equations as written above in their local form, but in addition, to avoid a substantial amount of algebra, the integrated form of the equations will also be used. Thus, defining mean quantities by

$$\bar{q} = \frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} q\,dn,$$

integration of (2.1*b*) followed by application of the boundary conditions (2.4) and (2.5) yields the result

$$\partial(T\overline{p}_{ss})/\partial s + KT\overline{p}_{ns} = 0, \tag{2.6a}$$

which represents a balance of forces parallel to the centre-line. A perpendicular balance can be found by integration of (2.1*c*):

$$\partial(T\overline{p}_{ss})/\partial s - KT\overline{p}_{ss} = 0. \tag{2.6b}$$

Finally, a balance of moments may be deduced by multiplying (2.1*b*) by n before integrating, so that

$$\partial M/\partial s - T\overline{p}_{ns} = 0, \tag{2.6c}$$

where the bending moment is

$$M = \int_{-\frac{1}{2}T}^{\frac{1}{2}T} dn\, np_{ss}.$$

An important equation that may be deduced from (2.6) is

$$\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial M}{\partial s}\right) + \frac{\partial M}{\partial s} = 0 \quad \left(\frac{\partial}{\partial\alpha} \equiv \frac{1}{K}\frac{\partial}{\partial s}\right), \tag{2.7}$$

which may be integrated to yield

$$\partial M/\partial s = A \sin\alpha + B\cos\alpha, \tag{2.8}$$

where A and B are unknown functions of time. We shall subsequently establish an independent functional relationship between M and α, so that (2.8) then becomes the central equation governing the problem.

The formulation of the problem is completed by a description of the kinematics. For our purposes it is sufficient to consider, in this respect, a viscida for which the velocity does not depend on n. A line element of fluid particles characterized at time t by $\delta\mathbf{s}$ and located at the centre-line remains on the centre-line during the motion and at time $t+\delta t$ is characterized by $\delta\mathbf{s}'$, where

$$\delta\mathbf{s}' = \delta\mathbf{s} + \delta t\,\delta\mathbf{q}. \tag{2.9}$$

Here, $\delta\mathbf{q}$ is the difference between the velocities of the ends of the element, so that

$$\delta\mathbf{q} = \delta s\,\partial\mathbf{q}/\partial s.$$

Taking the scalar product of $\delta\mathbf{s}'$ with itself, retaining only linear terms in δt, gives

$$\delta s' = \delta s\left[1 + \delta t\frac{\delta\mathbf{s}}{\delta s}\cdot\frac{\partial\mathbf{q}}{\partial s}\right],$$

in other words,

$$\delta s' = \delta s[1 + \delta t(\partial u/\partial s + vK)]. \tag{2.10}$$

This describes the elongation of the element. On the other hand, the cross-product of $\delta\mathbf{s}'$ with $\overline{\delta\mathbf{s}'}$ yields

$$-\mathbf{k}\frac{\delta\alpha}{\delta t} = \frac{\delta\mathbf{s}}{\delta s}\times\frac{\partial\mathbf{q}}{\partial s},$$

where $\delta\alpha$ is the change in inclination of the element and $\mathbf{k}$ is the unit normal to the plane containing the centre-line. Thus

$$\delta\alpha/\delta t = (-\partial v/\partial s + uK). \tag{2.11}$$

Results like (2.10) and (2.11) are familiar in the theory of elasticity when the deformation is $\mathbf{q}\,\delta t$ (Love 1944).

Since the fluid is incompressible, the extension predicted by (2.10) implies that the thickness of an element of the viscida changes by an amount

$$\delta T/\delta t = -T(\partial u/\partial s + vK). \tag{2.12}$$

Moreover, since s is measured from the left-hand end of the viscida, we have the additional result

$$\frac{\delta}{\delta t} \equiv \frac{\partial}{\partial t} + \left[\int_0^s ds\left(\frac{\partial u}{\partial s} + vK\right)\right]\frac{\partial}{\partial s}. \tag{2.13}$$

The relevance of these kinematic results stems from the fact that, when the viscida is very thin, the leading term of v is independent of n, and the leading term of u depends, at most, linearly on n. This linear dependence of u merely causes a shearing distortion of a viscida element and does not affect its rotation or thickening rate. Thus these quantities are correctly given, to *leading order*, by (2.11)–(2.13).

The equations deduced in this section [(2.6)–(2.8) and (2.11)–(2.13)] are all global equations, and by themselves can not be used to deduce the motion of the viscida. There is, of course, a close analogy between the equations of linear elasticity and those of Stokes flow, so that, since for the elastica the bending moment M is proportional to the curvature $\partial\alpha/\partial s$, we can infer for the present problem that M is proportional to the rate of change of curvature,

$$\frac{\delta}{\delta t}\left(\frac{\partial \alpha}{\partial s}\right).$$

Moreover, an elastica whose centre-line has an $O(1)$ deflexion does not change its length when it bends, and the equivalent result here (when K is $O(1)$) is that $\partial u/\partial s + vK$ is $o(1)$. These two additional results close the system, and detailed analysis of (2.3) is not necessary when K is $O(1)$. However, when K is $O(\epsilon)$, the elongation question is a more subtle one and detailed analysis can not be avoided. This is described in §3.

3. Analysis when the curvature $K = O(\epsilon)$

3.1. *Derivation of the governing equation for α*

In this section a solution of the governing equations is obtained which is valid when the centre-line curvature is of the order of magnitude of the thickness. The problem is characterized by a velocity (the relative speed of the two ends of the viscida) and two lengths (one equal to the long dimension of the viscida, the other characteristic of its thickness). A system of units is chosen such that the viscida length and the relative speed of its ends are both $O(1)$. ϵ, the small parameter of the problem, is then defined such that the thickness is $O(\epsilon)$. The analysis of this section is then based on the expansions

$$\begin{aligned} v &\sim \epsilon^{-1}v_{-1} + v_0 + \ldots, & T &\sim \epsilon T_1 + \epsilon^2 T_2 + \ldots,\\ u &\sim u_0 + \epsilon u_1 + \ldots, & p &\sim p_0 + \epsilon p_1 + \ldots,\\ \alpha &\sim \epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots, & K &\sim \epsilon K_1 + \epsilon^2 K_2 + \ldots,\end{aligned}$$

$$n = \epsilon N, \quad t = \epsilon^2 \tau.$$

The subscripted variables are functions of s, N and the scaled time τ. Starting the expansion for v with an $O(\epsilon^{-1})$ term implies that an $O(\epsilon^2)$ horizontal displacement of the ends gives rise to an $O(\epsilon)$ vertical displacement of the centre-line. Such a result would be expected for an inextensible strip, and is also appropriate for the viscida problem.

If the above expansions are substituted into the governing equations (2.3), a sequence of simpler equations arises, each of which may be integrated with respect to N. This integration introduces arbitrary functions of s, and certain constraints must be imposed on these functions in order to satisfy the boundary conditions (2.4) and (2.5). It is characteristic of problems of this type (e.g. Buckmaster 1973) that the emergence of the constraints is often delayed. For example a necessary constraint on $O(1)$ terms may only be uncovered when the $O(\epsilon)$ or $O(\epsilon^2)$ terms are examined.

The details of this procedure are straightforward, and only the following results will be needed:

$$v_{-1} = v_{-1}(s), \quad u_0 = u_{00}(s) + Nu_{01}(s) \tag{3.1), (3.2}$$

and

$$p_0 = p_{00}(s) + N(v''_{-1} - u'_{01}), \tag{3.3}$$

where

$$v'_{-1} + u_{01} = 0 \tag{3.4}$$

and

$$p_{00} + 2\mu(K_1 v_{-1} + u'_{00}) = 0. \tag{3.5}$$

The time dependence is implicit in these expressions.

Continuing with this procedure would eventually lead to a complete formulation of the problem, but use of the integrated equations makes this unnecessary. Thus the axial stress, as inferred from the above results, has the expansion

$$p_{ss} \sim 4\mu(u'_{00} + v_{-1}K_1 + u'_{01}N) + O(\epsilon) \tag{3.6}$$

and it follows that the bending movement may be written in the form

$$M \sim \epsilon^2 M_2 + O(\epsilon^3), \tag{3.7a}$$

where

$$M_2 = \tfrac{1}{3}\mu T_1^3 u'_{01} = -\tfrac{1}{3}\mu T_1^3 v''_{-1}. \tag{3.7b}$$

Furthermore, the leading velocity terms v_{-1} and u_0 are of the form for which the kinematic equations are justified, to leading order, so that (2.11)–(2.13) imply

$$\partial T_1/\partial\tau = 0, \quad \partial\alpha_1/\partial\tau = -v'_{-1}. \tag{3.8), (3.9}$$

Equations (3.7b) and (3.9) provide an additional relationship between M and α which enables (2.8) to be written in the form

$$\frac{\mu}{3}\frac{\partial}{\partial s}\left(T_1^3 \frac{\partial^2 \alpha_1}{\partial\tau\,\partial s}\right) = A(\tau)\,\alpha_1 + B(\tau). \tag{3.10}$$

Because of (3.8), the thickness T_1 is to be regarded as a given function of s.

Equation (3.10) is the fundamental equation governing the motion of the viscida. In order to solve it, it is appropriate to specify an initial distribution for α_1, together with boundary conditions at each end of the viscida. The analogous problem of the elastica suggests two archetypal problems, namely

$$\left.\begin{array}{l} \alpha_1 = 0 \quad (3.11a) \\ \text{or} \quad M_2 = 0 \quad (3.11b) \end{array}\right\} \text{at the ends.}$$

However, the second of these (the pinned-end problem) is physically not very realistic for our study so the discussion will be restricted to (3.11*a*).

Additional end conditions must be imposed since the functions $A(\tau)$ and $B(\tau)$ are unknown. These conditions describe the relative motion of the ends. If the left end is located at the origin of a fixed Cartesian frame, the right end has ordinate

$$y(L) = -\int_0^L \sin\alpha \, ds,$$

where L is the length of the thread. We make the choice $y(L) = 0$, so that α_1 must satisfy the constraint

$$\int_0^1 \alpha_1 ds = 0, \tag{3.12}$$

where, without loss of generality, the length has been assigned the value one. (Note that there can be no $O(1)$ change in L during the motion, because the time interval is $O(\epsilon^2)$.)

A second constraint follows from the specification of $x(L)$. First, however, note that from (2.6*c*) and (2.8) we may deduce the result

$$T_1(\overline{p}_{ns})_1 = A(\tau)\,\alpha_1 + B(\tau).$$

Differentiating with respect to α_1 and using (2.6*b*) then yields

$$A(\tau) = T_1(\overline{p}_{ss})_0,$$

whence, by virtue of (3.6),

$$A(\tau) = 4\mu T_1(u'_{00} + v_{-1}K_1). \tag{3.13}$$

Thus $A(\tau)$ is a measure of the local rate of extension of the viscida, as well as equalling the axial load.

Now

$$x(L) = \int_0^L \cos\alpha \, ds,$$

so that provided that α vanishes at the end points

$$\frac{dx}{dt}(L) = \frac{dL}{dt} - \int_0^L \sin\alpha \frac{\partial\alpha}{\partial t} ds.$$

But, from (2.10) and (3.13),

$$\frac{dL}{dt} \sim \int_0^L (u'_{00} + v_{-1}K_1)\, ds = \frac{A(\tau)}{4\mu}\int_0^L \frac{ds}{T_1},$$

so that with the choice $dx(L)/dt = -1$

$$1 = -\frac{A(\tau)}{4\mu}\int_0^1 \frac{ds}{T_1} + \int_0^1 \alpha_1 \frac{\partial\alpha_1}{\partial\tau} ds. \tag{3.14}$$

This constraint is appropriate when the ends of the viscida are moved towards each other with unit relative speed.

With α_1 prescribed as a function of s at $\tau = 0$, equations (3.10), (3.11*a*), (3.12) and (3.14) describe the evolution of α_1. This problem can be solved using eigenfunction expansions.

3.2. *Solution of the problem for* α_1

It is convenient to introduce a new time variable defined by

$$\tfrac{1}{3}\mu T_1^{*3}\, dm(\tau)/d\tau = -A(\tau), \quad m(0) = 0, \tag{3.15}$$

where T_1^* is a constant. The expansions

$$\alpha_1 \sim \sum_{n=0}^{\infty} D_n e^{m/\lambda_n} \phi_n(s), \quad \frac{B(\tau)}{A(\tau)} \sim -\sum_{n=0}^{\infty} D_n \frac{K_n}{\lambda_n} e^{m/\lambda_n}$$

then lead to the eigenvalue problem

$$\left.\begin{aligned} &\frac{d}{ds}\left(\frac{T_1^3}{T_1^{*3}}\frac{d\phi_n}{ds}\right) + \lambda_n \phi_n = K_n, \\ &\phi_n(0) = \phi_n(1) = 0, \quad \int_0^1 \phi_n ds = 0. \end{aligned}\right\} \tag{3.16}$$

This is not a Sturm–Liouville problem, but nevertheless the familiar arguments for establishing orthogonality, etc. (Courant & Hilbert 1962) also work here. Furthermore, an associated Sturm–Liouville problem may be defined;

$$\frac{d}{ds}\left(\frac{T_1^3}{T_1^{*3}}\frac{d\psi_n}{ds}\right) + \gamma_n \psi_n = 0,$$

$$\psi_n(0) = \psi_n(1) = 0.$$

Expanding each ϕ_n in terms of the ψ_n leads to useful results. In this way it is easy to show that for physically sensible choices of $T_1(s)$ an infinite set of discrete positive eigenvalues $\{\lambda_n\}$ and an infinite set of mutually orthogonal eigenfunctions $\{\phi_n\}$ are defined. Moreover,

$$\lim_{n\to\infty} (\lambda_n/n^2) = O(1) \tag{3.17}$$

and, in general, λ_n lies between γ_n and γ_{n+1}. If T_1 is symmetric about $s = \frac{1}{2}$, the antisymmetric (about $s = \frac{1}{2}$) eigenfunctions are identical to the antisymmetric Sturm–Liouville functions.

A special case, which can easily be solved, is that of the viscida of constant thickness, i.e.

$$T_1 = T_1^*.$$

The *symmetric* eigenfunctions are then

$$\phi_n = (K_n/\lambda_n)\,[1 - \cos\lambda_n^{\frac{1}{2}} s - \tfrac{1}{2}\lambda_n^{\frac{1}{2}} \sin\lambda_n^{\frac{1}{2}} s], \quad \tfrac{1}{2}\lambda_n^{\frac{1}{2}} = \tan\tfrac{1}{2}\lambda_n^{\frac{1}{2}}, \tag{3.18}$$

where K_n is only uniquely specified if we impose a normalization condition,

$$\int_0^1 \phi_n^2 ds = 1.$$

It is sufficient to note that, for large λ_n, K_n is then $O(\lambda_n^{\frac{1}{2}})$. The normalized *antisymmetric* eigenfunctions are

$$\phi_n = 2^{\frac{1}{2}} \sin 2n\pi s, \quad \lambda_n = 4n^2\pi^2, \quad K_n = 0. \tag{3.19}$$

Once the eigenfunctions are determined, the unknown function $m(\tau)$ can be found from the constraint (3.14), whence

$$1 = \frac{dm}{d\tau}(\tau)\left[\frac{T_1^{*3}}{12}\int_0^1 \frac{ds}{T_1} + \sum_n \frac{D_n^2}{\lambda_n} e^{2m/\lambda_n}\right]. \tag{3.20}$$

Further discussion of these results is facilitated by an estimate of the Fourier coefficients D_n, defined in terms of the initial centre-line deviation by

$$D_n = \int_0^1 \phi_n \alpha_1(s,0)\, ds. \tag{3.21}$$

It is reasonable, on physical grounds, to restrict ourselves to initial conditions that are fairly smooth. More precisely, we shall assume that $\partial^2\alpha_1(s,0)/\partial s^2$ is bounded on the interval. Moreover, for geometrical reasons, the initial data must satisfy the constraint (3.12). It is then an easy matter to show, by integrating (3.21) by parts twice, that for the viscida of constant thickness D_n is $O(n^{-2})$ or smaller when n is large. This estimate is also true in the general case provided that T_1 is sufficiently smooth. The infinite sum in (3.20) is then convergent, and for small τ,

$$m \sim \tau\left[\frac{T_1^{*3}}{12}\int_0^1 \frac{ds}{T_1} + \sum_n \frac{D_n^2}{\lambda_n}\right]^{-1}.$$

The initial axial load, which equals the end force needed to drive the motion, is then

$$\epsilon T_1(\bar{p}_{ss})_0 = -\tfrac{1}{3}\epsilon\mu T_1^{*3}\left[\frac{T_1^{*3}}{12}\int_0^1 \frac{ds}{T_1} + \sum_n \frac{D_n^2}{\lambda_n}\right]^{-1},$$

which is a decreasing function of the initial centre-line displacement.

The series in (3.20) may be written in the form

$$e^{2m/\lambda_1}\sum_n \frac{D_n^2}{\lambda_n}\exp\left[2m\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_1}\right)\right],$$

and this is absolutely convergent, uniformly so in $m \geqslant 0$. Consequently for large m,

$$\tau \sim \tfrac{1}{2}D_1^2 e^{2m/\lambda_1} + O(e^{2m/\lambda_2}).$$

An identical argument may be applied to the sum representing α_1, whence for large m,

$$\alpha_1 \sim D_1 e^{m/\lambda_1}\phi_1(s) + O(e^{m/\lambda_2}).$$

Combining these two results yields the asymptotic result

$$\alpha_1 \sim (2\tau)^{\frac{1}{2}}\phi_1(s). \tag{3.22}$$

Thus the first mode dominates the solution for large times, and its amplitude is independent of the prescribed amplitude at $\tau = 0$. In comparison the second mode has amplitude $\sim D_2(2\tau/D_1^2)^{\frac{1}{2}\lambda_1/\lambda_2}$. Of course if D_1 is zero this result has to be modified, and in general ϕ_1 in (3.22) has to be replaced by ϕ_j, where D_j is the first non-zero Fourier coefficient of the initial disturbance.

The problem is now seen to be one of stability. There is of course a solution corresponding to a straight viscida ($D_n = 0$), but this is an isolated solution in the sense that any initial disturbance, no matter how small, will grow in amplitude; in real life, the viscida will always buckle. The rate of growth of each mode of the disturbance depends on the corresponding eigenvalue. Eventually the lowest-order mode (that contributes to the initial disturbance) dominates. However, the other modes are only algebraically smaller, and it is conceivable that this dominance will not materialize, in practice, if there is an appropriate marked disparity between the magnitudes of the various D_n.

Since the solution we have obtained is unbounded, the assumptions on which the analysis of this section are based eventually break down. The 'inner expansion' developed here must, as $\tau \to \infty$, be replaced by an 'outer expansion'. This is the subject of the next section.

4. Analysis when the curvature $K = O(1)$

The analysis of this section closely follows that of § 3, except that it is based on the expansions

$$\begin{aligned} v &\sim v_0 + \epsilon v_1 + \dots, & T &\sim \epsilon T_1 + \dots, \\ u &\sim u_0 + \epsilon u_1 + \dots, & p &\sim p_0 + \epsilon p_1 + \dots, \\ \alpha &\sim \alpha_0 + \epsilon \alpha_1 + \dots, & K &\sim K_0 + \epsilon K_1 + \dots, \end{aligned}$$
$$n = \epsilon N,$$

where the subscripted variables are functions of s, N and t. The solution generated in this way can either be used to describe the evolution of an initial given $O(1)$ deviation of the centre-line, or be matched with the solution developed in § 3.

The results necessary to close the system of global equations derived in § 2 can, as mentioned earlier, be inferred from results for linear elasticity. Nevertheless, if the procedure of § 3 is followed we find

$$u_0 = u_0(s), \quad v_0 = -u_0'/K_0, \quad p_0 = 0,$$
$$u_1 = u_{10} + N u_{11}, \quad u_{11} = u_0 K_0 - v_0',$$
$$v = v_1(s),$$
$$p_1 = -2\mu(u_{10}' + K_0 v_1 + K_1 v_0) - 2\mu u_{11}' N \quad \text{etc.}$$

The second of these implies that there is no $O(1)$ elongation of the viscida, as expected. Furthermore $(p_{ss})_0$ vanishes, but

$$(p_{ss})_1 = 4\mu(u_{10}' + K_0 v_1 + K_1 v_0 + u_{11}' N),$$

whence the bending moment is

$$M \sim \tfrac{1}{3}\epsilon^3 \mu T_1^3 u_{11}' + \dots. \tag{4.1}$$

Since the leading velocity terms u_0 and v_0 do not depend on N, the kinematic equations of § 2 are applicable, and yield

$$\partial T_1/\partial t = 0, \tag{4.2a}$$
$$\partial \alpha_0/\partial t = u_0 K_0 - v_0' = u_{11}. \tag{4.2b}$$

Equations (4.1), (4.2) and (2.8) then lead to the equation that governs the evolution of the viscida shape, namely

$$\frac{\mu}{3}\frac{\partial}{\partial s}\left(T_1^3\frac{\partial^2\alpha_0}{\partial s\,\partial t}\right) = A(t)\sin\alpha_0 + B(t)\cos\alpha_0, \tag{4.3}$$

where T_1 is a given function of s.

Equation (4.3), when linearized for small values of α_0, is identical to (3.10), but the present problem for α_0 is not then identical to that of § 3 for α_1. Certainly we may choose, as before,

$$\alpha_0(0) = \alpha_0(1) = 0.$$

Also the nonlinear version of (3.12) is

$$\int_0^1 \sin\alpha_0\, ds = 0, \tag{4.4}$$

but the absence of elongation means that the equivalent of (3.14) is

$$1 = \int_0^1 \sin\alpha_0 \frac{\partial\alpha_0}{\partial t}\, ds. \tag{4.5}$$

It is m aningful to examine the solution of this system for small values of α_0. Suc h a solution can be used either to match with the solution of § 3, or else to de c be the early growth of an initially imposed, small (but $O(1)$) disturbance. W shall consider the latter problem first.

Since the l nearized version differs from the problem of § 3 only because the constraint (4.5), when linearized, differs from (3.14), the analysis is very similar and it is only the equation for $m(\tau)$ that is different. Indeed, this equation is [cf. (3.20)]

$$1 = \frac{dm}{dt}\sum_n \frac{D_n^2}{\lambda_n} e^{2m/\lambda_n} \tag{4.6}$$

and for small m,

$$m \sim t\left[\sum_n \frac{D_n^2}{\lambda_n}\right]^{-1}.$$

Since the bending moment is $O(\epsilon^3)$, it follows that the mean axial stress is $O(\epsilon^2)$ (i.e. $(\overline{p}_{ss})_1$ must vanish) and furthermore, the integrated equations imply that

$$A = T_1(\overline{p}_{ss})_2.$$

Therefore the initial end load is

$$\epsilon^2 T_1(\overline{p}_{ss})_2 = -\tfrac{1}{3}\epsilon^3\mu T_1^{*3}\left[\sum_n \frac{D_n^2}{\lambda_n}\right]^{-1}. \tag{4.7}$$

This should be compared with the $O(\epsilon)$ end load needed to drive the viscida when the curvature is $O(\epsilon)$. Equation (4.7) reflects this difference in that it is not valid when all the D_n vanish.

When m is large, (4.6) implies the asymptotic result

$$t \sim \tfrac{1}{2}D_1^2 e^{2m/\lambda_1},$$

so that

$$\alpha_0 \sim (2t)^{\frac{1}{2}}\phi_1(s). \tag{4.8}$$

Thus the linearization is only valid when $t^{\frac{1}{2}}$ is small, and this can be reconciled with the fact that m is large provided that D_1 is small enough. Just as in § 3, the first mode eventually dominates. Of course, when t is not small the linearization is invalid and recourse must be made to numerical methods.

The development of a solution that can be matched with the $O(\epsilon)$ solution developed in §3 is similar to the above, except that a slight redefinition of $m(t)$ is required, with

$$\lim_{t\to 0} m(t) = -\infty.$$

Only then can the eigenfunction expansion vanish as $t \to 0$. Now for large negative values of m, the high frequency end of the spectrum is dominant, so that only if the expansion for α_0 is truncated after a finite number of terms, that is

$$\alpha_0 = \sum_{n=1}^{j} D_n e^{m/\lambda_n} \phi_n(s) \tag{4.9a}$$

and

$$1 = \frac{dm}{dt} \sum_{n=1}^{j} \frac{D_n^2}{\lambda_n} e^{2m/\lambda_n}, \tag{4.9b}$$

can a match be made with the inner solution, which has the behaviour

$$\alpha_1 \sim (2\tau)^{\frac{1}{2}} \phi_j(s) \quad \text{as} \quad \tau \to \infty, \tag{4.10}$$

where, it may be recalled, ϕ_j is the lowest-order mode that contributes to the disturbance at $\tau = 0$. Then as $m \to -\infty$

$$t \sim \tfrac{1}{2} D_j^2 e^{2m/\lambda_j}, \quad \alpha_0 \sim (2t)^{\frac{1}{2}} \phi_j(s), \tag{4.11}$$

and matching to first order with the inner solution is assured. Note that none of the D_n are determined from the first-order matching; however it seems probable that $D_1, \ldots, D_{j-1}$ all vanish. Of course if $j = 1$ the solution (4.9) becomes

$$\alpha_0 = (2t)^{\frac{1}{2}} \phi_1(s). \tag{4.12}$$

Consequently, if the first mode dominates the $O(\epsilon)$ solution for large values of τ (as it will for arbitrary initial disturbances), it will continue to dominate for small but $O(1)$ values of t. Further evolution of the solution can only be described using numerical methods.

5. Numerical investigation

This section is concerned with the development of the solution of §4 for values of α_0 large enough to invalidate linearization. If the thickness of the viscida is constant the mathematical problem is to solve the equation

$$\partial^3\alpha_0/\partial s^2\, \partial t = F(t) \sin\alpha_0 + G(t) \cos\alpha_0 \tag{5.1}$$

subject to the constraints

$$\alpha_0(0,t) = \alpha_0(1,t) = 0, \tag{5.2a}$$

$$\int_0^1 \cos\alpha_0 \frac{\partial\alpha_0}{\partial t}\, ds = 0, \tag{5.2b}$$

$$\int_0^1 \sin\alpha_0 \frac{\partial\alpha_0}{\partial t}\, ds = 1. \tag{5.2c}$$

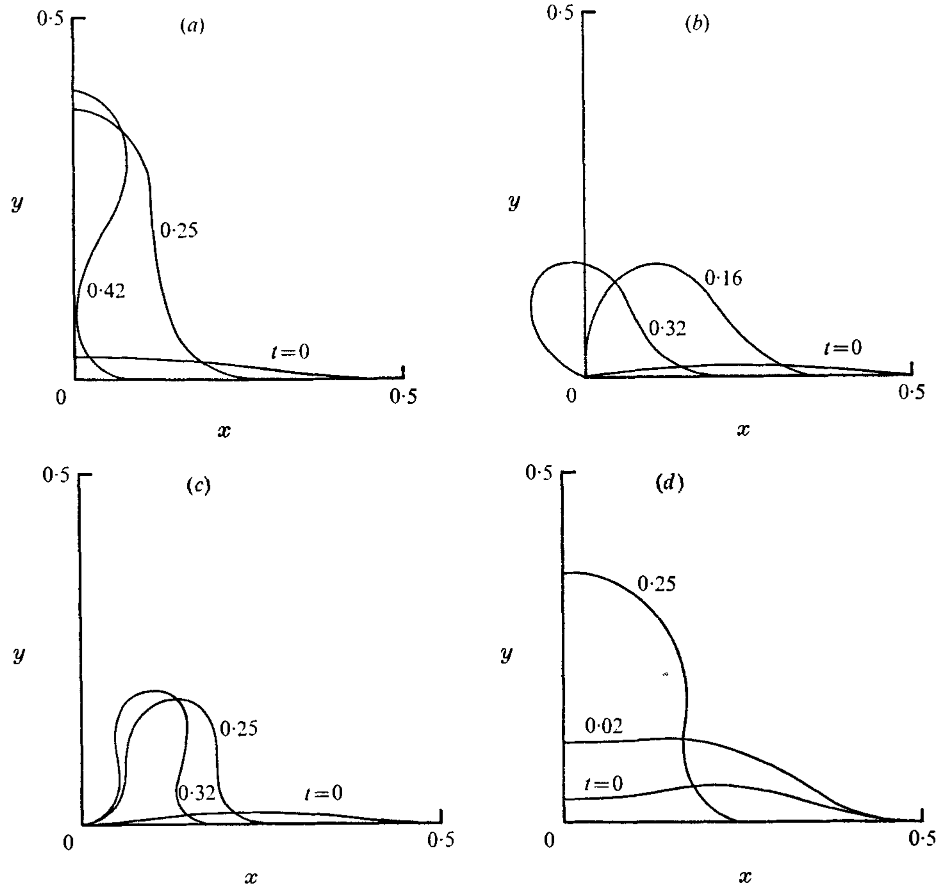

FIGURE 2. Buckled viscida. (*a*) First mode: $\alpha(s, 0) = -0{\cdot}1 \sin 2\pi s$. (*b*) Second mode: $\alpha(s, 0) = (0{\cdot}2/a)(\cos as - 1) + 0{\cdot}1 \sin as$. (*c*) Third mode: $\alpha(s, 0) = -0{\cdot}1 \sin 4\pi s$. (*d*) Mixed modes: $\alpha(s, 0) = -0{\cdot}1 \sin 2\pi s - 0{\cdot}2 \sin 4\pi s$.

Formally integrating (5.1), we have

$$\frac{\partial \alpha_0}{\partial t}(s, t) = F(t) \int_0^s ds' \int_0^{s'} \sin \alpha_0 \, ds'' + G(t) \int_0^s ds' \int_0^{s'} \cos \alpha_0 \, ds'' + H(t)\, s + I(t).$$

Thus if α_0 is a known function of s at any time, the constraints (5.2) may be used to determine the instantaneous values of F, G, H and I and this in turn leads to knowledge of $\partial \alpha / \partial t$ as a function of s. Changes in the configuration may then be determined by a forward integration in time. Figures 2(*a*)–(*d*) show some typical results of calculations of this kind. Figures 2(*a*)–(*c*) show the evolution of configurations which for small α_0 coincide, respectively, with the first three eigenfunctions described by (3.18)† and (3.19). The third mode appears to be of special interest since it compares favourably with the photographs reproduced in Taylor's (1969) paper. Other than noting at this time that Taylor's comparison with the third mode of buckling of a pinned end elastica appears to be inappropriate, we shall defer further discussion of these results until § 7.

† a is the smallest root of $\frac{1}{2}p = \tan \frac{1}{2}p$.

Figure 2 (*d*) shows the evolution of a disturbance that is initially a linear combination of the first two antisymmetric eigenfunctions. A configuration closely resembling the first mode quickly emerges, in agreement with the theoretical prediction.

6. The stretching problem

6.1. α *is* $O(1)$

We now want to consider the buckling problem in reverse. That is, at $t = 0$ the viscida has an arbitrary displacement and then the ends are pulled apart with an $O(1)$ velocity. Since the problem is kinematically reversible it might be thought that the stretching problem has no new features. However, an arbitrary displacement, as we have seen, cannot be generated by buckling from physically reasonable initial data; the displacement generated in this way is dominated by the low-order modes. Thus if an arbitrary displacement is smoothed by stretching pathological behaviour can be anticipated as the displacement vanishes.

Let us start by assuming that the displacement of the centre-line is $O(1)$. Then the formulation of the problem is identical to that of § 4 except that the constraint (4.5) is replaced by

$$-1 = \int_0^1 \sin\alpha_0 \frac{\partial\alpha_0}{\partial t}\, ds. \tag{6.1}$$

The change in sign reflects the fact that here the ends of the viscida are *pulled apart* with unit relative speed.

If α_0 is small enough to justify linearization, a solution can be constructed by eigenfunction expansions. The only difference from the earlier analyses is the equation satisfied by $m(t)$, which is now

$$-1 = \frac{dm}{dt}\Sigma\frac{D_n^2}{\lambda_n}e^{2m/\lambda_n}, \quad m(0) = 0. \tag{6.2}$$

Integration of this equation reveals that, as $m \to -\infty$, $t \to \frac{1}{2}\,\Sigma D_n^2$, so that after a finite time the viscida is straight to $O(1)$. The approach to the straight configuration is easily described if the initial disturbance is formed from only the first j eigenfunctions. For then, as $m \to -\infty$,

$$\alpha_0 \sim D_j e^{m/\lambda_j}\phi_j(s)$$

and

$$t - \tfrac{1}{2}\sum^{j} D_n^2 \sim -\tfrac{1}{2}D_j^2 e^{2m/\lambda_j},$$

whence

$$\alpha_0 \sim [\textstyle\sum^{j} D_n^2 - 2t]^{\frac{1}{2}}\,\phi_j(s). \tag{6.3}$$

Thus, as the centre-line deviation dies out, the solution is dominated by the highest-order mode, and is highly oscillatory if j is large.

If j is not finite the discussion is more difficult. A special case discussed briefly in the appendix suggests that the decay is then exponential in time, rather than algebraic like (6.3). Not surprisingly, the number of oscillations between $s = 0$ and $s = 1$ becomes unbounded as $t \to \frac{1}{2}\Sigma D_n^2$. This is the pathological behaviour hinted at earlier.

6.2. α *is* $O(\epsilon)$ *or smaller*

In the case when the initial deviation of the centreline is $O(\epsilon)$, the governing equations are those of § 3 except that (3.14) is replaced by

$$-1 = \frac{-A(\tau)}{4\mu}\int_0^1 \frac{ds}{T_1} + \int_0^1 \alpha_1 \frac{\partial \alpha_1}{\partial \tau}\, ds. \tag{6.4}$$

As a consequence, the equation defining $m(\tau)$ is

$$-1 = \frac{dm}{d\tau}(\tau)\left[\frac{T_1^{*3}}{12}\int_0^1 \frac{ds}{T_1} + \sum_n \frac{D_n^2}{\lambda_n} e^{2m/\lambda_n}\right], \quad m(0) = 0.$$

The disturbance dies out in the limit $m \to -\infty$ coincident with the asymptotic result

$$\tau \sim -m \frac{T_1^{*3}}{12}\int_0^1 \frac{ds}{T_1}. \tag{6.5}$$

The asymptotics for α_1 in terms of m are identical to those of § 6.1 for α_0; only when viewed as functions of time are there any differences. Thus if the highest-order mode that contributes to the disturbance is ϕ_j, we have the asymptotic result

$$\alpha_1 \sim D_j e^{m/\lambda_j} \phi_j(s)$$

and in view of (6.5) the final decay in time is exponential. This is to be compared with the algebraic behaviour of (6.3). If an infinite number of the modes make a contribution, the asymptotic result (A 6) (see appendix) is still appropriate, only now l is proportional to the time τ.

During the decay of the $O(\epsilon)$ disturbance, the rate of elongation of the thread plays an important role. However, there is no significant increase in length since the time interval is so small, being $O(\epsilon^2)$. Thus as $\tau \to \infty$ a new stage must emerge in the solution, in which the dominant motion is simply one of elongation. This elongation must occur over an $O(1)$ time interval. Clearly during this stage there is no $O(\epsilon)$ deviation of the centreline. In fact the above results strongly suggest that over most of the length of the viscida the curvature is exponentially small.

To analyse the stretching of a straight viscida we carry out an analysis analogous to that of § 3, but in a Cartesian co-ordinate system and based on the expansions

$$u \sim u_0 + \epsilon u_1 + \dots, \quad v \sim \epsilon v_1 + \dots, \quad \text{etc.},$$

where the subscripted variables are functions of x, $Y = y/\epsilon$ and t. It is easily shown that u_0 is independent of Y and is given by the expression

$$u_0 = C_1 \int_0^x \frac{dx}{T_1} + C_2,$$

where C_1 and C_2 are constants. Also,

$$v_1 = (-Y/T_1)\, C_1.$$

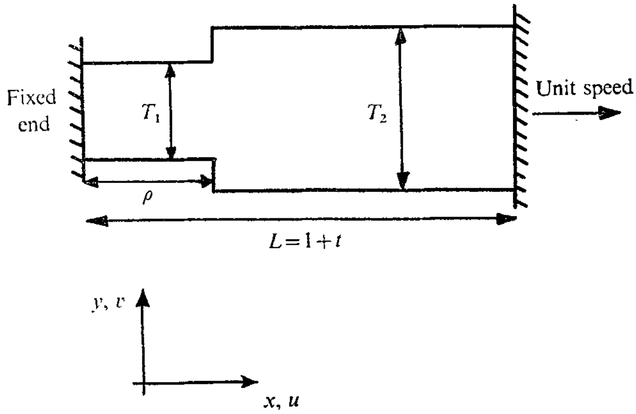

FIGURE 3. Straight stretched viscida.

If the left end ($x = 0$) is fixed and the right end ($x = L$) is moved to the right with unit speed we find that

$$u_0 = \int_0^x \frac{dx}{T_1} \Big/ \int_0^L \frac{dx}{T_1}. \tag{6.6}$$

This is simply a statement that the mean axial force is continuous. In addition we have the kinematic condition (2.12), which reduces to

$$\frac{\partial T_1}{\partial t} + u_0 \frac{\partial T_1}{\partial x} + \frac{\partial u_0}{\partial x} T_1 = 0. \tag{6.7}$$

If T_1 is initially independent of x, an assumption that we adopted to discuss the asymptotics of this section, it remains x independent and the viscida deforms as a rectangle. However, this final stage of the motion is going to occur in the general case, so that it is of interest to discuss the general features of (6.6) and (6.7). Let us consider the simple but illuminating case of a straight viscida divided into two sections of constant thickness (figure 3). Such a viscida, when stretched, will maintain its piecewise constant-thickness shape, and what is of interest is the manner in which the two thicknesses T_1 and T_2 change with time. The total length of the viscida is $1+t$, $t > 0$, and the first section, which has thickness T_1, has length $\rho(t)$. With the left end fixed, the result (6.6) implies that

$$\frac{d\rho}{dt} = \int_0^{\rho(t)} \frac{dx}{T_1} \Big/ \left[\int_0^{\rho(t)} \frac{dx}{T_1} + \int_{\rho(t)}^{1+t} \frac{dx}{T_2} \right]. \tag{6.8}$$

It is more convenient to deal with the fractional length $\zeta(t)$ of the first section defined by

$$\zeta(t) = \rho(t)/(1+t)$$

for this satisfies the equation

$$\frac{dt}{1+t} = d\zeta \frac{(\zeta+k)}{\zeta(1-\zeta)}, \quad \text{where} \quad k = \frac{T_1}{T_2 - T_1}.$$

Now the volume V_i of each section will be conserved during the motion, i.e.

$$(1+t)\,\zeta T_1 = V_1, \quad (1+t)(1-\zeta)\,T_2 = V_2.$$

This determines k in terms of ζ, whence

$$\frac{|\zeta - V_1/V|}{\zeta(1-\zeta)} = \frac{|\zeta(0) - V_1/V|}{\zeta(0)\,(1-\zeta(0))}(1+t), \tag{6.9}$$

where $V = V_1 + V_2$ is the total volume. It follows that if $T_1(0) > T_2(0)$, so that $\zeta(0) < V_1/V$, then ζ remains smaller than V_1/V during the motion, and for large times $\zeta \to 0$ (i.e. $T_2/T_1 \to 0$). On the other hand, if $T_1(0) < T_2(0)$ then for large times $\zeta \to 1$ (i.e. $T_1/T_2 \to 0$). In other words the thinner section stretches more rapidly. Since an arbitrary thickness distribution can be approximated by a string of rectangular sections, this conclusion is true in general. This is a 'necking' instability.

7. Concluding remarks and comparison with the elastica problem

The work described in this paper was partly motivated by a simple experiment of Taylor's (1969) in which the ends of a thin thread of highly viscous fluid were pushed together. One of our primary aims was to see if there are any favoured shapes associated with such a procedure. In this connexion we have shown that the first mode will emerge in a dominant role from an arbitrary sum of appropriate eigenfunctions. This first mode displays only one zero for α, apart from those at the two ends (e.g. figure 2 (a)). The only photographs reproduced by Taylor show deformed shapes which closely resemble the *third* mode (figure 2c). One possible explanation for this is that the first two modes made no significant contribution to the initial displacement of Taylor's thread. Recall in this connexion that the higher modes are only *algebraically* smaller, so that if D_3 is significantly larger than D_1 and D_2 it is conceivable that the third mode will dominate. It would, perhaps, be worth repeating his experiment. We plan to do this and shall briefly report the results in a second paper which will be primarily concerned with surface-tension effects.

A second possible explanation for the discrepancy is that Taylor may have rotated the ends as he pushed them together. Certainly it is possible to generate shapes comparable with those he observed by appropriate relaxation of the end conditions used in the present paper.

Taylor makes an explicit comparison between the experimental result and the third mode of buckling of a pinned-end elastica. A much more convincing visual comparison can be made with the third mode for a clamped-end elastica, and it is worth noting the similarities between that problem and the present one. $O(1)$ deformations of an elastica are governed by the equations

$$\left.\begin{aligned} \frac{d}{ds}\left(T^3\frac{d\alpha}{ds}\right) &= N\sin\alpha + M\cos\alpha, \\ \alpha(0) &= \alpha(1) = 0, \\ y(1) - y(0) &= -\int_0^1 \sin\alpha\, ds = 0, \\ x(1) - x(0) &= \int_0^1 \cos\alpha\, ds = 1 - \delta, \end{aligned}\right\} \tag{7.1}$$

where δ is given, and the only essential difference between these equations and those of §4 is the absence of the time derivative from the left side of (7.1). This is a direct consequence of the analogy between the equations of linear elasticity and those of Stokes flow. It is clear that the buckling modes (eigenfunctions) of the linearized problem (α small) are identical to the ϕ_j of §4. However, when α is not small, the difference between the equations plays a fundamental role and the 'natural' shapes of the viscida (the nonlinear continuation of the eigenfunctions) are only qualitatively similar to the elastica shapes.

One question that might arise in comparing the present results with Taylor's experiment is whether there is any fundamental difference between the two-dimensional situation analysed here and the real three-dimensional problem. Nachman (1973), in his thesis, has examined certain aspects of the problem for a viscida of circular cross-section whose centre-line moves in a plane. The result analogous to (4.2a) is that the cross-section remains circular, and the subsequent problem for α_0 is identical to that of §4. In the absence of surface tension three-dimensional effects do not appear to play any significant role.

Helpful discussions with G. S. S. Ludford are gratefully acknowledged. Part of this work was supported by AFOSR Grant AFOSR 73-2497.

Appendix

Consider the stretching of a viscida whose small but $O(1)$ displacement can be expressed as a sum of the antisymmetric eigenfunctions

$$\alpha_0 = 2^{\frac{1}{2}} \sum_{n=1}^{\infty} D_n \exp(m/4n^2\pi^2) \sin 2n\pi s, \tag{A 1}$$

where
$$D_n = 2^{\frac{1}{2}} \int_0^1 \alpha_0(s, 0) \sin 2n\pi s \, ds. \tag{A 2}$$

It is clear that, as $m \to -\infty$, α_0 vanishes slower than e^{km} for any k, and the number of oscillations between $s = 0$ and $s = 1$ is unbounded. Furthermore, for large values of $-m$ an arbitrary finite number of terms may be omitted from the sum (A 1) with only an exponentially small error. Thus the asymptotic behaviour may be deduced by replacing the Fourier coefficients D_n by their asymptotic representation for large n. If the initial disturbance is infinitely differentiable in the open interval $(0, 1)$ the leading contribution to the Fourier coefficients for large n arises from the fact that the derivatives of the associated periodic function are discontinuous at integer values of s. Thus integrating (A 2) by parts yields

$$D_n \sim \frac{2}{(2n\pi)^3} \left[\frac{\partial^2 \alpha_0}{\partial s^2}(1, 0) - \frac{\partial^2 \alpha_0}{\partial s^2}(0, 0) \right]$$

with an error that is no larger than $O(n^{-4})$ if the first four derivatives of $\alpha_0(s, 0)$ are bounded. Then the leading term in the asymptotic expansion of α_0 is identical with the leading term in the expansion of

$$f = C \sum_{n=1}^{\infty} n^{-3} e^{-l/n^2} \sin 2n\pi s, \tag{A 3}$$

where
$$C = \frac{1}{4\pi^3}\left[\frac{\partial^2\alpha_0}{\partial s^2}(1,0) - \frac{\partial^2\alpha_0}{\partial s^2}(0,0)\right]$$
and
$$l = -m/4\pi^2.$$

If we take the Laplace transform of f with respect to l and sum the series, we find
$$f = \frac{-C}{2\pi i}\int_{\mathrm{Br}} dp\, e^{pl}\left[\frac{\pi}{2}\frac{\sinh\left[(\frac{1}{2}-s)\,2\pi/p^{\frac{1}{2}}\right]}{\sinh(\pi/p^{\frac{1}{2}})} - \pi(s-\tfrac{1}{2})\right]. \tag{A 4}$$
The only singularities of the integrand are simple poles on the negative real axis that accumulate at the origin, and the asymptotic behaviour for large l will be deduced by a method based on an idea of G. S. S. Ludford, for which we express our gratitude.

If the inversion contour is partly bent back around the negative real axis, the constant may be omitted from the integrand and the denominator may be replaced by
$$\operatorname{cosech}(\pi/p^{\frac{1}{2}}) = 2\exp(-\pi/p^{\frac{1}{2}})\sum_{n=0}^{\infty}\exp(-2\pi n/p^{\frac{1}{2}})$$
since $\operatorname{Re} p^{\frac{1}{2}} > 0$. Then, writing $\sigma = l^{\frac{2}{3}}p$, we find
$$f = \frac{iC}{4}\sum_{n=0}^{\infty}\int_{\mathrm{Br}} d\sigma\, l^{-\frac{2}{3}}\left\{\exp\left[l^{\frac{1}{3}}\left(\sigma - \frac{2\pi}{\sigma^{\frac{1}{2}}}(n+s)\right)\right] - \exp\left[l^{\frac{1}{3}}\left(\sigma - \frac{2\pi}{\sigma^{\frac{1}{2}}}(n+1+s)\right)\right]\right\}. \tag{A 5}$$
This is a sum of integrals of the form
$$I_k = \int_{\mathrm{Br}} d\sigma \exp\left[l^{\frac{1}{3}}(\sigma - k/\sigma^{\frac{1}{2}})\right],$$
where k is a positive real constant, and each of these may be asymptotically evaluated using the method of steepest descent. The steepest-descent path through the saddle point
$$\sigma = (\tfrac{1}{2}k)^{\frac{2}{3}}\, e^{\frac{2}{3}i\pi}$$
starts at $-\infty + \frac{3}{2}i(\frac{1}{2}k)^{\frac{2}{3}}\, 3^{\frac{1}{2}}$, passes through the saddle point and then crosses the positive imaginary axis before reversing itself and approaching the origin in a direction that is parallel to the real axis, in the limit. Its mirror image in the real axis is the steepest-descent path through the saddle point at
$$\sigma = (\tfrac{1}{2}k)^{\frac{2}{3}}\, e^{-\frac{2}{3}i\pi}.$$

The union of these two paths is a suitable inversion path for the evaluation of I_k, and the dominant contribution, for large l, comes from the immediate vicinity of the two saddle points. In this way we deduce the asymptotic result
$$I_k \sim \frac{-i\pi^{\frac{1}{2}}\,16}{3^{\frac{1}{2}}\,8^{\frac{1}{2}}\,2^{\frac{5}{6}}}\, k^{\frac{1}{3}}\, l^{-\frac{1}{6}} \exp\left[-\tfrac{3}{4}\times 2^{\frac{1}{3}}\, l^{\frac{1}{3}}\, k^{\frac{2}{3}}\right] \sin\left[\frac{5\pi}{6} + \frac{3^{\frac{3}{2}}}{4}\, 2^{\frac{1}{3}}\, l^{\frac{1}{3}}\, k^{\frac{2}{3}}\right].$$
It is clear that the leading contribution to f comes from the term with the smallest value of k. Thus if $0 < s < \frac{1}{2}$,
$$\alpha_0 \sim \frac{C\pi 4}{3^{\frac{1}{2}}\,8^{\frac{1}{2}}\,2^{\frac{5}{6}}}\, k_1^{\frac{1}{3}}\, l^{-\frac{5}{6}} \exp\left[-\tfrac{3}{4}\times 2^{\frac{1}{3}}\, l^{\frac{1}{3}}\, k_1^{\frac{2}{3}}\right] \sin\left[\frac{5\pi}{6} + \frac{3^{\frac{3}{2}}}{4}\, 2^{\frac{1}{3}}\, l^{\frac{1}{3}}\, k_1^{\frac{2}{3}}\right], \tag{A 6}$$

where $k_1 = 2\pi s$. The symmetry condition implicit in (A 1) extends the result to the whole interval. There are regions of non-uniformity at $s = 0, \frac{1}{2}$ and 1 but we do not discuss them.

In order to complete the asymptotic description it is necessary to deduce the asymptotic behaviour of $m(t)$ from (6.2). We have

$$\Sigma D_n^2 - 2t = \Sigma D_n^2 e^{2m/\lambda_n}$$

and, still restricting ourselves to initial disturbances that are sums of the anti-symmetric eigenfunctions, we note that the right-hand side is asymptotically proportional to

$$g = \Sigma \frac{1}{n^6} e^{-2l/n^2}.$$

Proceeding in a way very similar to the discussion of f, an integral representation for g may be found, namely

$$g = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\pi}{2} \int_{\mathrm{Br}} dp\, e^{p2l} p^{\frac{3}{2}} \left[\exp\left(\frac{-2\pi}{p^{\frac{1}{2}}} n\right) + \exp\left(\frac{-2\pi}{p^{\frac{1}{2}}} (n+1)\right)\right],$$

where the Bromwich contour has been bent towards $\mathrm{Re}\, p = -\infty$ to ensure convergence. The asymptotic analysis then yields the result

$$g \sim \tfrac{3}{8}\pi^{\frac{1}{2}} (2l)^{-\frac{5}{2}} \quad \text{as} \quad l \to \infty,$$

so that

$$l \sim \text{constant}\, (\Sigma D_n^2 - 2t)^{-\frac{2}{5}}. \tag{A 7}$$

Combining this result with (A 6), we may conclude that over most of the length of the viscida the approach to the straight configuration is exponentially rapid. This may be compared with the algebraic behaviour of (6.3), valid when the eigen-function expansion is truncated. During the approach, the viscida becomes highly crinkled.

The present analysis can be extended to include the symmetric eigenfunctions but we have not done so. Our sole aim was to give some concrete indication of what can happen when the eigenfunction expansion is not truncated.

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